Lecture 16: Recall:

Prop: Let V be an inner product space and $S_n = \{\vec{w}_1, \ldots, \vec{w}_n\}$ be a linearly independent subset of V. Define:

$$S_n' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \text{ where } \vec{v}_1 = \vec{w}_1 \text{ and}$$

for $k = 2, \ldots, n$,

$$\vec{v}_k \overset{\text{def}}{=} \vec{w}_k - \sum_{j=1}^{k-1} \left(\frac{\langle \vec{w}_k, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \right) \vec{v}_j$$

Then: S_n' is orthogonal and $\operatorname{Span}(S_n') = \operatorname{span}(S_n)$

The above construction of an orthogonal basis is called

<u>Gram-Schmidt process.</u>

Consider $V = P(\mathbb{R})$ equipped with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt .$$

Let $\beta = \{1, x, x^2, \dots, x^n, \dots\}$ be standard ordered basis for $P(\mathbb{R})$.

Take $\vec{v}_1 = 1$.

Then: $$\vec{v}_2 = x - \frac{\overbrace{\langle x, \vec{v}_1 \rangle}^{=0}}{\underbrace{\| \vec{v}_1 \|^2}_{=2}} \overbrace{\vec{v}_1}^{=1} = x$$

$$\vec{v}_3 = x^2 - \frac{\overbrace{\langle x^2, \vec{v}_1 \rangle}^{=2/3}}{\underbrace{\| \vec{v}_1 \|^2}_{2}} \vec{v}_1 - \frac{\overbrace{\langle x^2, \vec{v}_2 \rangle}^{=0}}{\| \vec{v}_2 \|^2} \vec{v}_2 = x^2 - \frac{1}{3}$$

✓

$$\vec{v}_4 = x^3 - \cancelto{0}{\frac{\langle x^3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}} \vec{v}_1 \boxed{- \frac{\langle x^3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2} - \cancelto{0}{\frac{\langle x^3, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}} \vec{v}_3$$

$$= x^3 - \frac{3}{5} x$$ and so on, ...

for $P(\mathbb{R})$

This produces an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots\}$, whose elements are called <u>Legendre polynomial</u>.

Corollary: Let V be a non-zero finite-dim inner product space.

Then, V has an orthonormal basis $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ s.t.

$\forall \vec{x} \in V$, we have: $\vec{x} = \sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i$

Corollary: Let V be a non-zero finite-dim inner product space with an orthonormal basis $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Let T be a linear operator on V. Let $A = [T]_\beta$. Then: $A_{ij} = \langle T(\vec{v}_j), \vec{v}_i \rangle$.

Proof:

$$[T]_\beta = \begin{pmatrix} \cdots & \Big| \\ & [T(\vec{v}_j)]_\beta & \cdots \\ & \Big| \end{pmatrix}$$

$$T(\vec{v}_j) = \sum_{i=1}^{n} \underbrace{\langle T(\vec{v}_j), \vec{v}_i \rangle}_{= A_{ij}} \vec{v}_i$$

Orthogonal complement

Def: Let S be a non-empty subset of an inner product space V. The orthogonal complement of S is defined as:

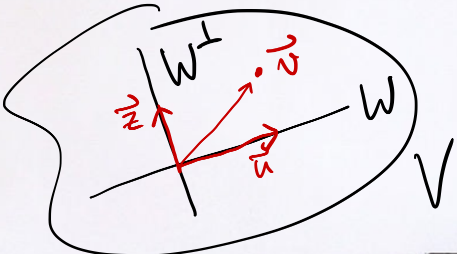
$$S^{\perp} \overset{\text{def}}{=} \{ \vec{x} \in V : \langle \vec{x}, \vec{y} \rangle = 0 \text{ for } \forall \vec{y} \in S \}$$

Proposition: Let V be an inner product space and $W \subset V$ a finite-dim subspace of V. Then: $\forall \vec{y} \in V$, $\exists !\, \vec{u} \in W$ and $\vec{z} \in W^{\perp}$ such that $\vec{y} = \vec{u} + \vec{z}$.

Furthermore, if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthonormal basis for W, then:

$$\vec{u} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

The vector $\vec{u} \in W$ is called the orthogonal projection of $\vec{y}$ on W.



Proof: Given $\vec{y} \in V$, we set $\vec{u} \stackrel{\text{def}}{=} \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i \in W$

and $\vec{z} \stackrel{\text{def}}{=} \vec{y} - \vec{u}$. Then: $\vec{y} = \underset{W}{\vec{u}} + \underset{W^{\perp}\,??}{\vec{z}}$.

Now, $\langle \vec{z}, \vec{v}_j \rangle = \langle \vec{y} - \vec{u}, \vec{v}_j \rangle = \langle \vec{y}, \vec{v}_j \rangle - \langle \vec{u}, \vec{v}_j \rangle$

$$= \langle \vec{y}, \vec{v}_j \rangle - \underbrace{\sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \langle \vec{v}_i, \vec{v}_j \rangle}_{\langle \vec{y}, \vec{v}_j \rangle}$$

$$\therefore \langle \vec{z}, \overbrace{\sum_{i=1}^{k} b_k \vec{v}_i}^{\in W} \rangle = \sum_{i=1}^{k} \overline{b_k} \cancelto{0}{\langle \vec{z}, \vec{v}_i \rangle} = 0 = 0$$

$\therefore \vec{z} \in W^{\perp}$

For uniqueness, suppose $\exists \vec{u}' \in W$ and $\vec{z}' \in W^\perp$ such that:

$$\vec{y} = \underset{\substack{\uparrow \\ W}}{\vec{u}} + \underset{\substack{\uparrow \\ W^\perp}}{\vec{z}} = \underset{\substack{\uparrow \\ W}}{\vec{u}'} + \underset{\substack{\uparrow \\ W^\perp}}{\vec{z}'} \Rightarrow \underset{\substack{\uparrow \\ W}}{\vec{u} - \vec{u}'} = \underset{\substack{\uparrow \\ W^\perp}}{\vec{z}' - \vec{z}} \in W \cap W^\perp$$

<u>Claim:</u> $W \cap W^\perp = \{\vec{0}\}$

<u>Pf:</u> Take $\vec{w} \in W \cap W^\perp$. Then: $\langle \underset{\substack{\uparrow \\ W}}{\vec{w}}, \underset{\substack{\uparrow \\ W^\perp}}{\vec{w}} \rangle = 0$

$$\Longleftrightarrow \vec{w} = \vec{0}$$

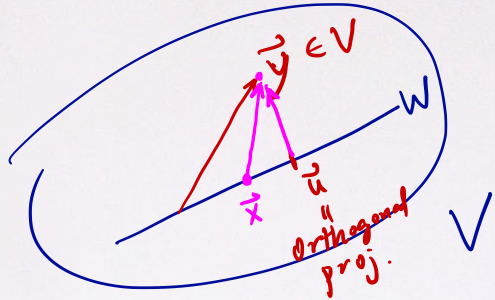
This implies: $\vec{u} - \vec{u}' = \vec{z}' - \vec{z} = \vec{0}$

$$\Longleftrightarrow \vec{u} = \vec{u}' \text{ and } \vec{z} = \vec{z}'.$$

Corollary: With notations as above, then :

$$\| \underset{V}{\vec{y}} - \underset{W}{\vec{x}} \| \geq \| \underset{V}{\vec{y}} - \underset{W}{\vec{u}} \| \quad \text{for } \forall \vec{x} \in W$$

and equality holds iff $\vec{x} = \vec{u}$



Remark: Orthogonal projection is the vector in W closest to $\vec{y}$.

Proof: Let $\vec{x} \in W$. Then: $\vec{y} = \underset{W}{\underset{\uparrow}{\vec{u}}} + \underset{W^{\perp}}{\underset{\uparrow}{\vec{z}}} \Rightarrow \vec{z} = \vec{y} - \vec{u}$

$$\|\vec{y} - \vec{x}\|^2 = \|\vec{u} + \vec{z} - \vec{x}\|^2 = \langle (\underset{W}{\underset{\uparrow}{\vec{u}}} - \underset{W}{\underset{\uparrow}{\vec{x}}}) + \underset{W^{\perp}}{\underset{\uparrow}{\vec{z}}}, (\underset{W}{\underset{\uparrow}{\vec{u}}} - \underset{W}{\underset{\uparrow}{\vec{x}}}) + \underset{W^{\perp}}{\underset{\uparrow}{\vec{z}}} \rangle$$

$$= \langle \vec{u} - \vec{x}, \vec{u} - \vec{x} \rangle + \underbrace{\langle \underset{W}{\underset{\uparrow}{\vec{u} - \vec{x}}}, \underset{W^{\perp}}{\underset{\uparrow}{\vec{z}}} \rangle}_{=0} + \underbrace{\langle \vec{z}, \vec{u} - \vec{x} \rangle}_{=0} + \langle \vec{z}, \vec{z} \rangle$$

$$= \|\vec{u} - \vec{x}\|^2 + \|\vec{z}\|^2 \geq \|\vec{z}\|^2 = \|\vec{y} - \vec{u}\|^2$$

The equality holds iff $\|\vec{u} - \vec{x}\|^2 = 0$ iff $\vec{u} = \vec{x}$.

Proposition: Suppose $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is an orthonormal set in an n-dimensional inner product space V. Then:

(a) S can extended to an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for V.

(b) If $W = \text{span}(S)$, then $S_1 = \{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthonormal basis for $W^\perp$ (exercise)

(c) If W is any subspace of V, then:

$$\dim(V) = \dim(W) + \dim(W^\perp)$$

Proof: (a) We first extend S to a basis:

$$\{\overbrace{\underbrace{\vec{v}_1, \ldots, \vec{v}_k}_{\text{L. I.}}}^{S}, \vec{w}_{k+1}, \ldots, \vec{w}_n\} \text{ for } V.$$

Then, we apply the G-S process to this basis.

$\because$ S is orthonormal, $\therefore$ $\vec{v}_1, \ldots, \vec{v}_k$ remains the same during the G-S process.

So, this process gives an orthonormal basis for V of the form

$$\{\underbrace{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k}_{\text{unchanged}}, \underbrace{\vec{v}_{k+1}, \ldots, \vec{v}_n}_{\text{new}}\}$$

(c) For any W, choose an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ for W and extend it to an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V.

Then:

$$\dim(V) = n = k + (n-k)$$
$$= \dim(W) + \dim(W^\perp)$$

Remark: In fact, a "complement" to a subspace $W \subset V$ $(\dim(V) < \infty)$ is another subspace $U \subset V$ s.t.

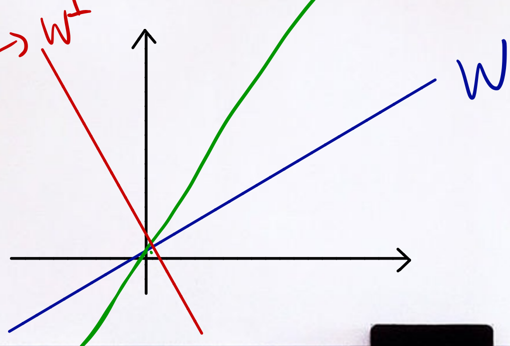
$$\begin{cases} \cdot\ W \cap U = \{\vec{0}\} \\ \cdot\ \dim(W) + \dim(U) = \dim(V) \end{cases} \implies V = W \oplus U$$

$$\Downarrow$$

$$V = W + U$$

$U \leftarrow$ is a complement to W

a special complement to W $\rightarrow W^{\perp}$



Adjoint of a linear operator

Prop: Let V be a finite-dim. inner product space over F.

Then for any linear transformation $g: V \to F$ (linear functional),

$\exists ! \, \vec{y} \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$.

Proof: Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis for V.

Set: $\vec{y} = \sum_{i=1}^{n} \overline{g(\vec{v}_i)} \, \vec{v}_i$.

We have: $\langle \vec{v}_j, \vec{y} \rangle = \sum_{i=1}^{n} g(\vec{v}_i) \langle \vec{v}_j, \vec{v}_i \rangle = g(\vec{v}_j)$

$\Rightarrow \quad g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$

If $\exists\, \vec{y}\,' \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y}\,' \rangle$ for $\forall\, \vec{x}$.

then, $\langle \vec{x}, \vec{y} \rangle = g(\vec{x}) = \langle \vec{x}, \vec{y}\,' \rangle$ for $\forall\, \vec{x}$

$$\Rightarrow \quad \vec{y} = \vec{y}\,'.$$

Theorem: Let V be a finite-dim inner product space. Let T be a linear operator on V. Then: $\exists!$ linear operator $T^*: V \to V$ such that: $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle$ for $\forall \vec{x}, \vec{y} \in V$.

T^* is called the adjoint of T.

Proof: Given any $\vec{y} \in V$, the map $g_{\vec{y}}: V \to F$ defined by.

$g_{\vec{y}}(\vec{x}) = \langle T(\vec{x}), \vec{y} \rangle$ is linear ($\because$ $\langle \cdot, \cdot \rangle$ is linear in the 1^{st} argument) + T is linear

By the previous proposition, $\exists! \, \vec{y}\,' \in V$ such that $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, \vec{y}\,' \rangle$ for all $\vec{x} \in V$.

$\overset{\displaystyle ''}{g_{\vec{y}}(\vec{x})}$

Now, we define uniquely: $T^*: V \to V$ by $T^*(\vec{y}) = \vec{y}\,'$.

To see that T^* is linear, let $\vec{y}_1, \vec{y}_2 \in V$ and $c \in F$.

Then $\forall \vec{x} \in V$, we have:

$$
\begin{aligned}
\langle \vec{x}, T^*(c\vec{y}_1 + \vec{y}_2) \rangle &= \langle T(\vec{x}), c\vec{y}_1 + \vec{y}_2 \rangle \\
&= \bar{c} \langle T(\vec{x}), \vec{y}_1 \rangle + \langle T(\vec{x}), \vec{y}_2 \rangle \\
&= \bar{c} \langle \vec{x}, T^*(\vec{y}_1) \rangle + \langle \vec{x}, T^*(\vec{y}_2) \rangle \\
&= \langle \vec{x}, c\, T^*(\vec{y}_1) + T^*(\vec{y}_2) \rangle
\end{aligned}
$$

$$\Rightarrow \quad T^*(c\vec{y}_1 + \vec{y}_2) = c\,T^*(\vec{y}_1) + T^*(\vec{y}_2)$$

Remark:

$$\langle \vec{x}, T(\vec{y}) \rangle = \overline{\langle T(\vec{y}), \vec{x} \rangle} = \overline{\langle \vec{y}, T^*(\vec{x}) \rangle} = \langle T^*(\vec{x}), \vec{y} \rangle$$

Proposition: Let V be a finite-dim inner product space and let β be an orthonormal basis for V. Then $\forall\, T: V \to V$, we have:

$$[T^*]_\beta = \left([T]_\beta\right)^* \leftarrow \text{conjugate tranpose}$$

$$\left(A^* = \left(\overline{A}\right)^T \right)$$

Proof: Let $A = [T]_\beta$, $B = [T^*]_\beta$ and $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

$$B_{ij} = \langle T^*(\vec{v}_j), \vec{v}_i \rangle = \langle \vec{v}_j, T(\vec{v}_i) \rangle = \overline{\langle T(\vec{v}_i), \vec{v}_j \rangle}$$

$$= \overline{A_{ji}}$$

//